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Multiplet Structures of BPS Solitons¹

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ABSTRACT

There exist simple single-charge and multi-charge BPS p -brane solutions in the D -dimensional maximal supergravities. From these, one can fill out orbits in the charge vector space by acting with the global symmetry groups. We give a classification of these orbits, and the associated cosets that parameterise them.

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1 Introduction

The BPS p -brane solutions in supergravity theories play a central rôle in the understanding of the non-perturbative structures of M-theory and string theory. For this reason, it is useful to develop a framework for characterising these solutions, and in particular, for describing their multiplet structures under the duality symmetries of the theories. In this paper, we shall be principally concerned with studying the multiplet structures of BPS solitons in maximal supergravities. In particular, we shall study the orbits of the global supergravity symmetry groups E_{11-D} in D dimensions [1, 2],¹ to see how they fill out multiplets of solutions.

A systematic way to study the orbits is first to consider the simplest single-charge BPS p -brane solutions, which preserve $\frac{1}{2}$ of the supersymmetry. Such a solution is described by a single harmonic function in the space transverse to the world-volume of the p -brane. Acting with the global symmetry group G , one obtains more complicated solutions in which, typically, more than one charge is non-vanishing. Of course since the global symmetry transformations G commute with supersymmetry, all the solutions in the orbit will also preserve $\frac{1}{2}$ of the supersymmetry. All the solutions in the orbit are characterised by the same single harmonic function. In some cases these orbits fill out the entire charge vector-space. Under these circumstances, all points in the charge vector space are associated with BPS solutions that preserve $\frac{1}{2}$ the supersymmetry.

In other cases, it turns out that the orbits of the original single-charge solution fill out only a subspace in the entire charge vector space. The signal for this is that there can be points in the charge space, in the neighbourhood of the single-charge starting point, which do not lie on the orbit. When this occurs, it turns out that there exists a “simple” 2-charge solution, which has both the original charge and any one of the charges that lies outside the infinitesimal orbit of the original charge. Here, by a simple 2-charge solution, we mean one that is characterised by two independent harmonic functions, associated with exactly two non-vanishing charges.² All such 2-charge solutions preserve $\frac{1}{4}$ of the supersymmetry.

¹In all cases, unless otherwise indicated, it is to be understood that the groups that we encounter will be of the maximally non-compact form, $E_{n(n)}$, *etc.* We shall in general omit the explicit indication of the maximal non-compact form.

²In this paper, we shall use the term “simple N -charge solution” to mean a solution with N independent harmonic functions that are associated with exactly N non-vanishing charges. Such solutions form multiplets under the Weyl group of the global symmetry group G [3]; they were classified in [4]. Acting with G on a simple N -charge solution, one always obtains configurations with $N' \geq N$ non-vanishing charges [4]. Equality is achieved only for Weyl-group elements.

One can now repeat the exercise of filling out the orbits under G , taking the simple 2-charge solution as a new starting point. Again, there is the possibility that the orbits now cover the entire charge vector space, in which case the multiplet analysis for this class of solution is complete. Alternatively, it may be the case that there are still points in the neighbourhood of the simple 2-charge solution that cannot be reached by the orbit. This indicates the existence of a simple 3-charge solution. By iteratively applying this procedure of adding in further “missed” charges, one eventually finds orbits that do cover the entire charge space. It turns out that for solutions whose charges are carried by the 4-form field strength, simple 1-charge solutions are always sufficient to cover the entire charge vector space. For solutions supported by 3-forms and 2-forms, the maximal values of N for simple N -charge solutions are dimension dependent; they were all obtained in [5,6], and classified in [4]. For 3-forms, one always has $N \leq 2$, and for 2-forms, $N \leq 4$. The different orbits can also be characterised by certain group-invariant polynomials, as discussed in [7].

It should be emphasised that the problem that we are addressing in this paper, of studying the action of the global symmetry groups on the charge vectors in BPS solutions, is not the same as the problem of studying the spectrum-generating symmetries for BPS solutions. To give the spectrum of BPS solitons, one needs to classify the sets of solutions at *fixed* values of the scalar moduli, *i.e.* the asymptotic values of all the dilatonic and axionic scalars. Although the standard global symmetry algebras are effective in mapping between different points in the charge vector space, they also in general change the scalar moduli at the same time. This problem was addressed in [9], in the case of the spectrum obtained from single-charge BPS solutions that preserve $\frac{1}{2}$ of the supersymmetry. Since the spectrum for fixed moduli necessarily involves solutions with different masses, it is manifest that the standard global symmetry groups cannot generate the complete spectrum. As was shown in [9], the additional ingredient of the overall scaling symmetry that every supergravity theory possesses is needed also, in order to fill out the entire spectrum. (The scaling transformation, called the “trombone” transformation in [9], is a symmetry of the equations of motion, corresponding to an homogeneous scaling of the action.) It turns out that the spectrum generating symmetry for the solutions that preserve $\frac{1}{2}$ the supersymmetry is in fact the same group as the global symmetry group G , but now realised non-linearly on the fields. We shall return to a discussion of this point in section 9, and in particular address the question of whether one can expect the results in [9] to extend to the multi-charge solutions that preserve less than $\frac{1}{2}$ of the supersymmetry.

In this paper, we shall study the orbits of the charge vectors for all the p -branes in

$D \geq 4$ that are supported by field strengths of degree 4, 3 and 2. All of these field strengths form linear representations under the global symmetry groups. In all cases, the dimensions of the global symmetry groups are larger than the dimensions of the charge vector spaces. In other words, there is a stability subgroup K of G that leaves the initial simple N -charge configuration fixed. The orbits are therefore parameterised by points in the coset G/K . We shall determine these coset structures for all the above p -brane orbits. (We shall focus on the classical coset structure in this paper. At the quantum level, the continuous global symmetry groups will be discretised to the U-duality groups discussed in [8].) The coset for the type IIB string doublet was obtained in [9], and the cosets for $D = 5$ and $D = 4$ have recently been obtained in [11].

In order to describe the maximal supergravities in D dimensions, we shall use the notation and conventions of [5]. The D -dimensional Lagrangian is given by

$$\begin{aligned} \mathcal{L} = & eR - \frac{1}{2}e(\partial\vec{\phi})^2 - \frac{1}{48}e\vec{a}\cdot\vec{\phi}F_4^2 - \frac{1}{12}e\sum_i e^{\vec{a}_i\cdot\vec{\phi}}(F_3^{(i)})^2 - \frac{1}{4}e\sum_{i<j} e^{\vec{a}_{ij}\cdot\vec{\phi}}(F_2^{(ij)})^2 \\ & - \frac{1}{4}e\sum_i e^{\vec{b}_i\cdot\vec{\phi}}(\mathcal{F}_2^{(i)})^2 - \frac{1}{2}e\sum_{i<j<k} e^{\vec{a}_{ijk}\cdot\vec{\phi}}(F_1^{(ijk)})^2 - \frac{1}{2}e\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}}(\mathcal{F}_1^{(ij)})^2 + \mathcal{L}_{FFA} , \end{aligned} \quad (1)$$

where the subscripts on the various field strengths indicate their degrees, with F_4 , $F_3^{(i)}$, $F_2^{(ij)}$ and $F_1^{(ijk)}$ coming from the 4-form in $D = 11$, while $\mathcal{F}_2^{(i)}$ and $\mathcal{F}_1^{(ij)}$ come from the dimensional reduction of the metric. The “dilaton vectors” \vec{a} , \vec{a}_i , \vec{a}_{ij} , \vec{a}_{ijk} , \vec{b}_i and \vec{b}_{ij} are constant $(11-D)$ -component vectors that characterise the couplings of the dilatonic scalars $\vec{\phi}$ to the various field strengths. Their detailed forms, and also their scalar products, are all given in [5]:

	F_{MNPQ}	vielbein
4 – form :	$\vec{a} = -\vec{g}$,	
3 – forms :	$\vec{a}_i = \vec{f}_i - \vec{g}$,	
2 – forms :	$\vec{a}_{ij} = \vec{f}_i + \vec{f}_j - \vec{g}$,	$\vec{b}_i = -\vec{f}_i$,
1 – forms :	$\vec{a}_{ijk} = \vec{f}_i + \vec{f}_j + \vec{f}_k - \vec{g}$,	$\vec{b}_{ij} = -\vec{f}_i + \vec{f}_j$.

(2)

The explicit expressions for the vectors \vec{g} and \vec{f}_i , which have $(11-D)$ components in D dimensions, are given in [5]. For our purposes, it is sufficient to note that they satisfy the relations

$$\vec{g} \cdot \vec{g} = \frac{2(11-D)}{D-2}, \quad \vec{g} \cdot \vec{f}_i = \frac{6}{D-2}, \quad \vec{f}_i \cdot \vec{f}_j = 2\delta_{ij} + \frac{2}{D-2}. \quad (3)$$

The scalar sector of the D -dimensional theory comprises the $(11-D)$ dilatons $\vec{\phi}$, and the axions $A_0^{(ijk)}$ and $\mathcal{A}_0^{(ij)}$ whose field strengths are the 1-forms $F_1^{(ijk)}$ and $\mathcal{F}_1^{(ij)}$. In addition,

in the conventionally “fully-dualised” versions of the theories, where all field strengths of degree $> \frac{1}{2}D$ are dualised, there will be additional axions in $D \leq 5$ associated with the dualisation of $(D - 1)$ -form field strengths. Their corresponding dilaton vectors will be the negatives of those for the fields prior to dualisation. Including the axions coming from dualisation, the total number of scalars is equal to the dimension of the Borel subgroup of the E_{11-D} Cremmer-Julia global symmetry group [12]. In fact, the dilatons $\vec{\phi}$ are in one-to-one correspondence with the Cartan subalgebra, while the axions are in one-to-one correspondence with the positive roots of E_{11-D} . In particular, their dilaton vectors are precisely the positive root vectors. The simple roots in D dimensions may be taken to be $\vec{b}_{i,i+1}$ and \vec{a}_{123} , for $1 \leq i \leq 10 - D$. It is easy to verify from (2) and (3) that these vectors have the dot products of the simple roots of E_{11-D} , as shown in Table 1.

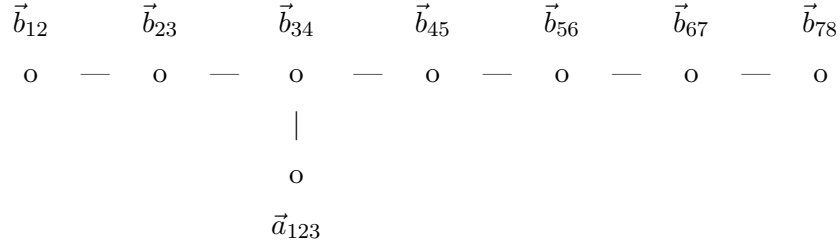


Table 1: The dilaton vectors $\vec{b}_{i,i+1}$ and \vec{a}_{123} generate the E_{11-D} Dynkin diagram

In each dimension D , the diagram is truncated to the part that survives when only the simple roots with indices $i \leq 11 - D$ are retained.

In the fully-dualised theories, the higher-degree fields form linear highest-weight representations under the E_{11-D} group. The associated dilaton vectors are the weight vectors of the representation [3]. The action of the generators of the global symmetry group on the states in the various representations can easily be determined by means of the dilaton vectors. The dilaton vectors for the axions are the positive roots of the E_{11-D} algebra (in the fully-dualised theory). The group generator associated with a root $\vec{\alpha}$ gives a non-vanishing result when acting on a state of weight \vec{w} if and only if $\vec{w} + \vec{\alpha}$ is also a weight in the representation. We shall make repeated use of this in what follows. All of the necessary results can be derived from the information about the dilaton vectors that is contained in (2) and (3).

The paper is organised as follows. Sections 2 to 7 contain the discussion of the charge vector orbits for field strengths with degrees ≥ 2 in all dimensions between $D = 10$ and $D = 4$. In each case, we analyse all the multi-charge starting points that are needed in order to fill out the orbits that span the entire charge spaces, and we discuss the supersymmetry of the various classes of solution. Our results in $D = 5$ and $D = 4$ are similar to those obtained in [11]. In section 8, we consider the extension of these ideas to solutions where the charges are carried by 1-form field strengths. Owing to the non-linearity of the action of the global symmetry groups on these fields, a general discussion is much more complicated than in the higher-degree cases. Instead, we consider a simplified situation where a subset of the axions form a linear representation under a subgroup of the full global symmetry group. The paper ends with concluding remarks in section 9. A summary of the results for the various cosets is contained in an appendix.

2 $D = 10$ Type IIB and $D = 9$

In both of these cases, there is an $SL(2, \mathbb{R})$ global symmetry. Let us first consider the type IIB theory. This contains two 3-form field strengths, which transform as a doublet under $SL(2, \mathbb{R})$, and a singlet self-dual 5-form, together with the dilaton and axion that parameterise the $SL(2, \mathbb{R})/SO(2)$ scalar manifold. The BPS string solutions, with electric charges (Q_1, Q_2) for the two 3-form field strengths, form a single multiplet under $SL(2, \mathbb{R})$. For example, one can start from the solution with charges $(1, 0)$, and fill out the entire 2-dimensional charge space by acting with $SL(2, \mathbb{R})$.

In fact, we should pause at this point to make this statement more precise. At the same time as the standard $SL(2, \mathbb{R})$ symmetry transforms the charges, it also changes the scalar moduli, *i.e.* the asymptotic values of the dilaton and axion. What we are really interested in is to fill out the 2-dimensional charge space at *fixed* values of the moduli. In particular, this involves BPS solutions with different masses, and it is manifest that the standard $SL(2, \mathbb{R})$ symmetry cannot by itself achieve this, since it leaves the metric invariant and hence preserves the mass. As was recently discussed in [9], the true spectrum-generating symmetry that fills out the 2-dimensional charge space at fixed values of the scalar moduli is obtained by including also the scaling symmetry of the theory, which of course *does* change the mass. To be precise, the charge space can be generated by acting with the denominator subgroup (which is $SO(2)$ in this case) to rotate the original charge vector $(1, 0)$ keeping its length fixed, and then acting with the rescaling “trombone” symmetry

to arrive at the desired radius in the 2-dimensional charge plane. An alternative way to describe this, which allows a proper treatment of the effects of Dirac quantisation in the quantum case, is that we act with the standard $SL(2, \mathbb{R})$ on the charge space, and then apply compensating symmetry transformations that leave the charges fixed, but restore the scalar moduli to their former values. Specifically, this is done by acting with a combination of a transformation in the Borel subgroup of $SL(2, \mathbb{R})$, together with a trombone rescaling. The Borel transformation³ allows us to restore the scalar moduli to their original values, but at the price that the charges are rescaled (although their ratio remains fixed). The trombone transformation then restores the scale of the charges, while leaving the moduli at their proper restored values. The net effect of this is that we now have a *non-linear* realisation of $SL(2, \mathbb{R})$, which acts as a true spectrum generating symmetry [9]. In the quantum case, this spectrum-generating $SL(2, \mathbb{R})$ is discretised to $SL(2, \mathbb{Z})$, in order to ensure that the Dirac quantisation condition is satisfied.

The results in [9] established that the group of the true spectrum-generating symmetry is the same, *qua* abstract group, as the standard Cremmer-Julia symmetry group, but now realised non-linearly on the fields. However, its realisation on the charges is still linear, and so for our present purposes we may continue to think of the standard linearly-realised Cremmer-Julia group as the group that fills out multiplets in the vector space of the charges, with the understanding that the full description of the associated BPS solutions would require compensating transformations on the scalar moduli. We shall return to a discussion of this point in section 9.

The group-theoretic structure of the spectrum of the type IIB BPS string solitons was discussed in [9]. The $SL(2, \mathbb{R})$ transformations

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \quad (4)$$

do not all act effectively on the initial charge vector $(1, 0)$; in fact the strict Borel subgroup

$$B_{\text{strict}} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \quad (5)$$

leaves the initial charge vector $(1, 0)$ fixed. Therefore the points in the 2-dimensional charge space are parameterised by elements of the coset $SL(2, \mathbb{R})/B_{\text{strict}}$. Clearly matrices of the form (5) give a realisation of \mathbb{R} , and so we may equivalently say that the charge space for

³The Iwasawa decomposition of the E_{11-D} global symmetry groups into the product of the Borel and maximal compact subgroups has been studied in [10].

the BPS strings in the type IIB theory is parameterised by [9]

$$\frac{SL(2, \mathbb{R})}{\mathbb{R}}. \quad (6)$$

The initial single-charge solution, and the solutions corresponding to all other points in the charge space, preserve $\frac{1}{2}$ of the supersymmetry.

An analogous discussion can be given for the multiplet of 5-branes, corresponding to the case where the two 3-form field strengths carry magnetic charges. These charges also transform as a doublet under $SL(2, \mathbb{R})$, contragrediently with respect to the transformations of the electric charges. Thus again, the charge space is parameterised by the coset $SL(2, \mathbb{R})/\mathbb{R}$, and again all points are associated with BPS solutions that preserve $\frac{1}{2}$ of the supersymmetry.

There also exist self-dual 3-branes in the type IIB theory, supported by the self-dual 5-form. This is a singlet under $SL(2, \mathbb{R})$. However, we know that BPS 3-brane solitons can exist for arbitrary values of the charge (classically speaking). These solutions can all be obtained from a solution with a given charge, by acting with the trombone scaling transformation. (Such transformations are also responsible for the M-brane solutions in $D = 11$ occurring for arbitrary values of the charge.)

Turning now to maximal supergravity in nine dimensions, we now have a $GL(2, \mathbb{R}) = SL(2, \mathbb{R}) \times \mathbb{R}$ global symmetry (together also with a trombone scaling symmetry). There is an $SL(2, \mathbb{R})$ doublet of 3-form field strengths $F_3^{(i)}$, and an $SL(2, \mathbb{R})$ doublet of 2-form field strengths $\mathcal{F}_2^{(i)}$. In addition, there is a singlet 4-form F_4 , a singlet 2-form $F_2^{(12)}$, two dilatonic scalars and an axionic scalar. Here, we shall look at the group structure of the various BPS solitons under the $SL(2, \mathbb{R})$ group.

Since the 4-form is a singlet under $SL(2, \mathbb{R})$, the only way to get an arbitrary-charge solution from a given one is by means of the trombone symmetry. The analysis of the spectrum of BPS strings or 4-branes for a doublet of 3-forms $F_3^{(i)}$ is identical to that in the type IIB case, and again the 2-dimensional charge space is parameterised by the coset $SL(2, \mathbb{R})/\mathbb{R}$.

The discussion for the 2-form solutions is slightly more complicated. There are three 2-forms in total, namely the $SL(2, \mathbb{R})$ doublet $\mathcal{F}_2^{(i)}$ and the singlet $F_2^{(12)}$. If we start from a single-charge solution using one member of the doublet $\mathcal{F}_2^{(i)}$, then acting with $SL(2, \mathbb{R})$ can only fill out the 2-dimensional plane $(p_1, p_2, 0)$ in the 3-dimensional charge space (p_1, p_2, u_{12}) . However, there also exist 2-charge solutions using the field strengths $\{\mathcal{F}_2^{(1)}, F_2^{(12)}\}$ or $\{\mathcal{F}_2^{(2)}, F_2^{(12)}\}$. If, for example, we start from the charge configuration

$(1, 0, u_{12})$, where u_{12} is any fixed charge for $F_2^{(12)}$, then acting with $SL(2, \mathbb{R})$ will fill out the entire 2-plane (p_1, p_2, u_{12}) at the fixed value of u_{12} . Since u_{12} can be given an arbitrary value, it follows that any point in the 3-dimensional charge space is associated with a p -brane solution. Those with charges $(p_1, p_2, 0)$ or $(0, 0, u_{12})$ preserve $\frac{1}{2}$ the supersymmetry, whilst those with charges (p_1, p_2, u_{12}) with at least one of p_1 or p_2 non-zero preserve $\frac{1}{4}$.

We have seen in the above discussion that in certain cases, namely the 3-form solutions in type IIB or in $D = 9$, BPS solutions corresponding to any point in the 2-dimensional charge space are filled out by acting with the global symmetry group on a basic single-charge solution, which preserves $\frac{1}{2}$ of the supersymmetry. On the other hand, the full set of BPS 2-form solutions in $D = 9$ is filled out by starting with a basic 2-charge solution, and acting with the global symmetry group. Note that this 2-charge solution preserves $\frac{1}{4}$ of the supersymmetry, and the number of charges cannot be reduced to one by any global symmetry transformation. In fact, global symmetry transformations only yield solutions with $N \geq 2$ charges. For this reason, it is useful to define the basic 2-charge solution as being a *simple* 2-charge solution. The full classification of all simple N -charge p -branes in all dimensions $D \geq 2$ was given in [4].

3 $D = 8$

The global symmetry group in $D = 8$ is $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$, where the first factor has \vec{b}_{12} and \vec{b}_{23} as simple roots, and the second has the simple root \vec{a}_{123} (see Table 1). The full root system is given by $\pm \vec{b}_{ij}$ and $\pm \vec{a}_{123}$. There is one 4-form field strength F_4 , which, together with its dual, forms a doublet under the $SL(2, \mathbb{R})$. The three 3-forms $F_3^{(i)}$ form a triplet under $SL(3, \mathbb{R})$, and the six 2-forms $F_2^{(ij)}$ and $\mathcal{F}_2^{(i)}$ form a (3,2) irreducible representation under $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$. The associated dilaton vectors are the weight vectors of these various irreducible representations. They, and the highest-weight vectors, are presented in Table 2. Also included is the list of negative roots whose generators annihilate the highest-weight states (of course all the positive roots, by definition, annihilate the highest weights).

Degree	Weight vectors	Highest weight	Negative roots that annihilate highest weight
4-form	$\vec{a}, -\vec{a}$	$-\vec{a}$	$-\vec{b}_{ij}$
3-forms	\vec{a}_i	\vec{a}_3	$-\vec{b}_{12}, -\vec{a}_{123}$
2-forms	\vec{a}_{ij}, \vec{b}_i	\vec{a}_{23}	$-\vec{b}_{23}$

Table 2: Weights and highest weights in $D = 8$

Since the electric and magnetic charges of the 4-form form a doublet under $SL(2, \mathbb{R})$, the discussion of its multiplet structure is identical to that for the $SL(2, \mathbb{R})$ multiplets in the previous section. The points in the 2-dimensional electric/magnetic charge space are parameterised by $SL(2, \mathbb{R})/\mathbb{R}$. The \mathbb{R} denominator group is the strict Borel subgroup of the $SL(2, \mathbb{R})$, whose generator has the positive root \vec{a}_{123} .

The highest-weight state $|\vec{a}_3\rangle$ representing the single-charge 3-form solution is annihilated not only by the positive-root generators $E_{\vec{b}_{ij}}$ and $E_{\vec{a}_{123}}$, but also by $E_{-\vec{b}_{12}}$ and $E_{-\vec{a}_{123}}$. This means that only the negative-root generators $E_{-\vec{b}_{13}}$ and $E_{-\vec{b}_{23}}$ have a non-vanishing action on the highest-weight state, giving $|\vec{a}_1\rangle$ and $|\vec{a}_2\rangle$ respectively. In addition, there is one combination of Cartan generators under which the state $|\vec{a}_3\rangle$ has non-vanishing weight (the combination proportional to $\vec{a}_3 \cdot \vec{H}$). Thus there is a three-parameter family of motions of the highest-weight state. In other words, the coset space that parameterises the 3-form solutions obtained from the global group action on the single charge associated with $|\vec{a}_3\rangle$ is three dimensional. The stability subgroup is generated by $E_{\vec{b}_{ij}}$ and $E_{\vec{a}_{123}}$, together with $E_{-\vec{b}_{12}}$ and $E_{-\vec{a}_{123}}$ and the two remaining combinations of the Cartan generators,

$$\vec{H}' = \vec{H} - \frac{3}{4}(\vec{a}_3 \cdot \vec{H}) \vec{a}_3 . \quad (7)$$

The vectors $\pm\vec{a}_{123}$ and $\pm\vec{b}_{12}$ are the non-vanishing roots of two commuting $SL(2, \mathbb{R})$ groups, generated by $E_{\pm\vec{a}_{123}}$ and $E_{\pm\vec{b}_{12}}$, whose two Cartan generators are the projections of \vec{H}' onto the mutually-orthogonal directions \vec{a}_{123} and \vec{b}_{12} respectively (the roots of the two commuting $SL(2, \mathbb{R})$ groups are unchanged by the projection (7) onto \vec{H}' , since $\vec{a}_3 \cdot \vec{a}_{123} = \vec{a}_3 \cdot \vec{b}_{12} = 0$). The remaining generators, $E_{\vec{b}_{13}}$ and $E_{\vec{b}_{23}}$, are mutually commuting, and they commute with $E_{\pm\vec{a}_{123}}$, but form a doublet under the $SL(2, \mathbb{R})$ generated by $E_{\pm\vec{b}_{12}}$, with weight vectors

$$\pm \frac{1}{2}(\vec{b}_{13} + \vec{b}_{23}) \quad (8)$$

under \vec{H}' . Thus the structure of the 3-dimensional coset space parameterising the 3-charge solutions is

$$\frac{SL(3, \mathbb{R}) \times SL(2, \mathbb{R})}{(SL(2, \mathbb{R}) \ltimes \mathbb{R}^2) \times SL(2, \mathbb{R})} . \quad (9)$$

Since the $SL(2, \mathbb{R})$ factor generated by $E_{\pm \vec{a}_{123}}$ is common to the numerator and denominator, the coset reduces to

$$\frac{SL(3, \mathbb{R})}{SL(2, \mathbb{R}) \ltimes \mathbb{R}^2} , \quad (10)$$

where the $SL(2, \mathbb{R})$ is generated by $E_{\pm \vec{b}_{12}}$, with roots $\pm \vec{b}_{12}$, and the two commuting \mathbb{R} generators have weight vectors \vec{b}_{13} and \vec{b}_{23} . The fact that they transform as a doublet under $SL(2, \mathbb{R})$ is reflected in the semi-direct product symbol \ltimes .

Note that the dimension of the coset (10) is equal to the dimension of the space of 3-form charges, *i.e.* 3. This is not a coincidence; the 3-forms transform as the vector representation of $SL(3, \mathbb{R})$, and so any point in the charge space can be reached by acting with $SL(3, \mathbb{R})$ on a given charge, which we are taking to be associated with the highest-weight state $|\vec{a}_3\rangle$. Thus any point in the 3-dimensional charge space has an associated BPS string or 3-brane solution, which preserves $\frac{1}{2}$ of the supersymmetry.

Finally in this section, we consider the 2-form field strengths, which form a (3,2)-dimensional irreducible representation under $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$, with \vec{a}_{23} as the highest weight. First, we consider the multiplet filled out by acting on the single-charge solution associated with this highest-weight state. From Table 2, we see that $|\vec{a}_{23}\rangle$ is annihilated by $E_{\pm \vec{b}_{23}}$, which form the non-zero-root generators of an $SL(2, \mathbb{R})$, and by $E_{\vec{b}_{12}}$, $E_{\vec{b}_{13}}$ and $E_{\vec{a}_{123}}$. These are mutually-commuting generators, with $(E_{\vec{b}_{12}}, E_{\vec{b}_{13}})$ transforming as a doublet under the $SL(2, \mathbb{R})$, and $E_{\vec{a}_{123}}$ as a singlet. In addition, there are two combinations of the Cartan generators that leave \vec{a}_{23} invariant, namely

$$\vec{H}' = \vec{H} - \frac{3}{7}(\vec{a}_{23} \cdot \vec{H}) \vec{a}_{23} . \quad (11)$$

One combination is the Cartan generator $\vec{b}_{23} \cdot \vec{H}'$ in the $SL(2, \mathbb{R})$ subgroup generated by $E_{\pm \vec{b}_{23}}$, and the other, which commutes with every generator in the stability group, enlarges the $SL(2, \mathbb{R})$ to $GL(2, \mathbb{R})$. Thus the coset space that parameterises the orbits of the single-charge 2-form solutions is

$$\frac{SL(3, \mathbb{R}) \times SL(2, \mathbb{R})}{GL(2, \mathbb{R}) \ltimes \mathbb{R}^3} . \quad (12)$$

In fact it is easy to understand the structure of the denominator group; there is a $GL(2, \mathbb{R})$ global symmetry in $D = 9$, which clearly, since it leaves the $D = 9$ metric invariant, will also leave the new Kaluza-Klein vector $\mathcal{A}_1^{(3)}$ in $D = 8$ invariant. In addition, the three vector

potentials $\{\mathcal{A}_1^{(1)}, \mathcal{A}_1^{(2)}, \mathcal{A}_1^{(12)}\}$ in $D = 9$ give rise to three axions in $D = 8$, which have three commuting global \mathbb{R} symmetries (two of which form a doublet under $GL(2, \mathbb{R})$). Thus the full invariance group for a solution using the new Kaluza-Klein vector is $GL(2, \mathbb{R}) \ltimes \mathbb{R}^3$. Since $\mathcal{A}_1^{(3)}$ is equivalent to the previous potential $A_1^{(23)}$ associated with the highest-weight vector \vec{a}_{23} under the full global symmetry group $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$, it follows that the solution based on the field strength associated with $|\vec{a}_{23}\rangle$ will also have $GL(2, \mathbb{R}) \ltimes \mathbb{R}^3$ as its stability group.

The dimension of the coset space (12) is 4, while the dimension of the charge space for 2-forms is 6. This shows that not all points in the total charge space can be reached by acting with global symmetry transformations on a single-charge solution. In particular, if we start from the single charge associated with the state $|\vec{a}_{23}\rangle$, we will never generate the charges associated with the states $|\vec{b}_2\rangle$ and $|\vec{b}_3\rangle$. In fact, there also exist simple 2-charge solutions for 2-forms in $D = 8$, and we can choose to augment the original charge associated with \vec{a}_{23} by precisely one or other of the charges that the single-charge orbits have failed to cover. For example, we may consider the case where $F_2^{(23)}$ and $\mathcal{F}_2^{(3)}$ carry the charges [5]. Thus to reveal the structure of the coset space parameterising these solutions, we need to identify the simultaneous stability subgroup for the pair of states $|\vec{a}_{23}\rangle$ and $|\vec{b}_3\rangle$. To do this, it is useful to tabulate the action of the complete set of all the non-zero-root generators of $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$ on the two states. Labelling the states by their weights, we have:

Weight	\vec{b}_{12}	\vec{b}_{13}	\vec{b}_{23}	\vec{a}_{123}	$-\vec{b}_{12}$	$-\vec{b}_{13}$	$-\vec{b}_{23}$	$-\vec{a}_{123}$
\vec{a}_{23}	—	—	—	—	\vec{a}_{13}	\vec{a}_{12}	—	\vec{b}_1
\vec{b}_3	—	\vec{b}_1	\vec{b}_2	\vec{a}_{12}	—	—	—	—

Table 3: Action of generators on $|\vec{a}_{23}\rangle$ and $|\vec{b}_3\rangle$ in $D = 8$

We see from Table 3 that the generators $E_{\vec{b}_{12}}$ and $E_{-\vec{b}_{23}}$ leave both of the states $|\vec{a}_{23}\rangle$ and $|\vec{b}_3\rangle$ invariant. In addition, we see that the combinations of generators

$$L_+ = (E_{\vec{b}_{13}} - E_{-\vec{a}_{123}}), \quad L_- = (E_{-\vec{b}_{13}} - E_{\vec{a}_{123}}) \quad (13)$$

will leave the sum of states $|\vec{a}_{23}\rangle + |\vec{b}_3\rangle$ invariant. These form the positive and negative root generators of an $SL(2, \mathbb{R})$ algebra. The Cartan generator for this $SL(2, \mathbb{R})$ must come from the one combination of the three Cartan generators of the original $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$

under which both the states $|\vec{a}_{23}\rangle$ and $|\vec{b}_3\rangle$ have zero weight, namely

$$\vec{H}' = \vec{H} - \frac{1}{8}(5\vec{a}_{23} \cdot \vec{H} + 7\vec{b}_3 \cdot \vec{H})\vec{b}_3 - \frac{1}{8}(5\vec{b}_3 \cdot \vec{H} + 7\vec{a}_{23} \cdot \vec{H})\vec{a}_{23} . \quad (14)$$

(The specific coefficients that arise here in this Gramm-Schmidt projection follow from the details of the dot products of the dilaton vectors involved in the construction. These dot products can all be determined from the results given in (2) and (3).) Thus the new Cartan generator H' is the projection of the original 3-vector \vec{H} of Cartan generators onto the line orthogonal to the plane containing the weights \vec{a}_{23} and \vec{b}_3 associated with the two charges. Note that although the generators L_{\pm} are not eigenstates under the original Cartan generators \vec{H} , they are eigenstates under the projected generators \vec{H}' . After a calculation, one finds that the two $SL(2, \mathbb{R})$ generators L_{\pm} have weights $\pm \frac{1}{2}(\vec{b}_{13} - \vec{a}_{123})$ under H' . In total, we therefore have five generators that leave the two-charge combination invariant. Clearly L_+, L_- and the Cartan combination \vec{H}' form an $SL(2, \mathbb{R})$ algebra, and the mutually commuting pair $E_{\vec{b}_{12}}$ and $E_{-\vec{b}_{23}}$ form a doublet under this $SL(2, \mathbb{R})$. Thus the stability subgroup of the 2-charge combination is $SL(2, \mathbb{R}) \ltimes \mathbb{R}^2$, and the space of 2-charge solutions is parameterised by the coset

$$\frac{SL(3, \mathbb{R}) \times SL(2, \mathbb{R})}{SL(2, \mathbb{R}) \ltimes \mathbb{R}^2} . \quad (15)$$

Note that this space has dimension 6, which is the same as the total dimension of the space of 2-form charges in $D = 8$. In fact, this coset allows us to reach all points in the 6-dimensional charge space, beginning from a given 2-charge solution. This can be seen from the non-trivial entries in Table 3, which indicate that all six of the states corresponding to the six 2-forms can be reached from the 2-charge starting point. (The Cartan combinations $\vec{a}_{23} \cdot \vec{H}$ and $\vec{b}_3 \cdot \vec{H}$ allow the original 2 charges to be rescaled arbitrarily, and the various combinations of non-zero-root generators that do not annihilate $|\vec{a}_{23}\rangle$ or $|\vec{b}_3\rangle$ allow the other four charges to be turned on with arbitrary strengths.) In general, the BPS solutions will preserve $\frac{1}{4}$ of the supersymmetry, unless one or other of the original charges is zero, in which case the solution will preserve $\frac{1}{2}$ of the supersymmetry.

4 $D = 7$

The global symmetry group in $D = 7$ is $SL(5, \mathbb{R})$, with simple roots $\vec{b}_{12}, \vec{b}_{23}, \vec{b}_{34}$ and \vec{a}_{123} . The full root system is given by $\pm \vec{b}_{ij}$ and $\pm \vec{a}_{ijk}$, associated with the generators $E_{\pm \vec{b}_{ij}}$ and $E_{\pm \vec{a}_{ijk}}$. In the fully-dualised theory there are five 3-forms $F_3^{(i)}$ and $*F_4$, and ten 2-forms

$F_2^{(ij)}$ and $\mathcal{F}_2^{(i)}$. The dilaton vectors associated with the 3-forms and the 2-forms are the weight vectors of the 5 and the $\bar{10}$ of $SL(5, \mathbb{R})$ respectively.

Let us begin by considering the 3-forms. Here, the highest-weight vector is $-\vec{a}$, associated with the 3-form $*F_4$. It is easy to see the state $|-\vec{a}\rangle$ is annihilated not only by all the positive-root generators, but also by the negative-root generators associated with the roots $-\vec{b}_{ij}$, and by the three combinations of the $SL(5, \mathbb{R})$ Cartan generators that are orthogonal to \vec{a} ;

$$\begin{aligned} E_{\pm \vec{b}_{ij}} , \quad E_{\vec{a}_{ijk}} , \\ \vec{H}' = \vec{H} - \frac{5}{8}(\vec{a} \cdot \vec{H}) \vec{a} . \end{aligned} \quad (16)$$

On the other hand, the other negative-root generators $E_{-\vec{a}_{ijk}}$ act on $|-\vec{a}\rangle$ to give $|\vec{a}_\ell\rangle$, where i, j, k and ℓ are all different. The conjugate pairs $E_{\pm \vec{b}_{ij}}$, together with the three Cartan generators \vec{H}' , generate $SL(4, \mathbb{R})$. The four generators $E_{\vec{a}_{ijk}}$ mutually commute, and form the vector representation under $SL(4, \mathbb{R})$, with weight vectors $\vec{a}_{ijk} + \frac{5}{4}\vec{a}$ under \vec{H}' . Thus the stability group of a single 3-form charge in $D = 7$ is $SL(4, \mathbb{R}) \ltimes \mathbb{R}^4$, and the coset is

$$\frac{SL(5, \mathbb{R})}{SL(4, \mathbb{R}) \ltimes \mathbb{R}^4} . \quad (17)$$

The coset has dimension 5, which equals the dimension of the space of 3-form charges. Indeed, any point in this 5-dimensional vector space can be reached by acting on the original single charge vector with $SL(5, \mathbb{R})$. As usual, this can be seen by observing that all the states of the 5-dimensional representation for the 3-forms can be reached by the action of the generators of $SL(5, \mathbb{R})$ that do not annihilate the highest-weight state $|-\vec{a}\rangle$. The stability group in (17) can also be understood as follows: In the $D = 7$ Lagrangian obtained by direct dimensional reduction from $D = 11$ without any dualisation, the global symmetry group is $SL(4, \mathbb{R}) \ltimes \mathbb{R}^4$ [13, 12], under which the 4-form is a singlet.

Let us now consider the 2-forms in $D = 7$. The highest-weight vector is \vec{a}_{34} . It is easily verified that the following subset of the $SL(5, \mathbb{R})$ generators annihilate $|\vec{a}_{34}\rangle$:

$$\begin{aligned} E_{\pm \vec{b}_{12}} , \quad E_{\pm \vec{b}_{34}} , \quad E_{\pm \vec{a}_{123}} , \quad E_{\pm \vec{a}_{124}} , \\ \vec{H}' = \vec{H} - \frac{5}{12}(\vec{a}_{23} \cdot \vec{H}) \vec{a}_{23} , \\ E_{\vec{a}_{134}} , \quad E_{\vec{a}_{234}} , \quad E_{\vec{b}_{13}} , \quad E_{\vec{b}_{23}} , \quad E_{\vec{b}_{14}} , \quad E_{\vec{b}_{24}} . \end{aligned} \quad (18)$$

The conjugate pair $E_{\pm \vec{b}_{12}}$ together with the projection of \vec{H}' onto \vec{b}_{12} generate an $SL(2, \mathbb{R})$, while the remaining three conjugate pairs, and the remaining two components of \vec{H}' generate

an $SL(3, \mathbb{R})$. The remaining six generators in (18) mutually commute, and form a (3,2) under the $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$; the highest weight generator is $E_{\vec{a}_{234}}$, with weight vector $\vec{a}_{234} - \frac{5}{6}\vec{a}_{34}$ under \vec{H}' . The coset parameterising the single-charge 2-form solutions in $D = 7$ is therefore

$$\frac{SL(5, \mathbb{R})}{(SL(3, \mathbb{R}) \times SL(2, \mathbb{R})) \ltimes \mathbb{R}^6} . \quad (19)$$

The denominator group can also be easily understood from the viewpoint of dimensional reduction from $D = 8$ to $D = 7$. In $D = 8$, the global symmetry group is $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$, which leaves the new Kaluza-Klein vector invariant. The R^6 comes from the commuting shift symmetries of the six axions that are dimensional reduction of six vectors in $D = 8$, which form a (3,2)-dimensional representation under $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$.

The coset (19) for a simple single-charge orbit is 7-dimensional, and so it does not parameterise all possible charge points in the 10-dimensional vector space for 2-forms. In the infinitesimal neighbourhood of the charge vector for the field strength $F_2^{(34)}$, three charges, carried by field strengths $F_2^{(12)}$, $\mathcal{F}_2^{(3)}$ and $\mathcal{F}_2^{(4)}$, cannot be reached. As discussed in the introduction, when this happens we can construct a simple 2-charge solution, where the original charge is augmented by any of the above three charges. To cover the entire space, we must then start with a simple 2-charge configuration. Any simple 2-charge solution provides an equivalent starting point for generating the charge space, since they form a single multiplet under the Weyl group of the global symmetry group [3]. Thus without loss of generality, we shall consider a 2-charge solution constructed from the pair of field strengths $\{F_2^{(12)}, F_2^{(34)}\}$, corresponding to the states $\{|\vec{a}_{12}\rangle, |\vec{a}_{34}\rangle\}$. It is easy to verify that their sum is annihilated by the following $SL(5, \mathbb{R})$ generators

$$E_{\pm \vec{b}_{12}} , \quad E_{\pm \vec{b}_{34}} , \quad (E_{\pm \vec{b}_{24}} - E_{\mp \vec{b}_{13}}) , \quad (E_{\pm \vec{b}_{14}} - E_{\mp \vec{b}_{23}}) , \quad E_{\vec{a}_{ijk}} , \quad (20)$$

together with the two Cartan generators

$$\vec{H}' = \vec{H} - \frac{1}{4}(2\vec{a}_{34} \cdot \vec{H} + 3\vec{a}_{12} \cdot \vec{H}) \vec{a}_{12} - \frac{1}{4}(2\vec{a}_{12} \cdot \vec{H} + 3\vec{a}_{34} \cdot \vec{H}) \vec{a}_{34} . \quad (21)$$

The conjugate pairs of generators in (20) have the weights

$$\pm \vec{b}_{12} , \quad \pm \vec{b}_{34} , \quad \pm \frac{1}{2}(\vec{b}_{24} - \vec{b}_{13}) , \quad \pm \frac{1}{2}(\vec{b}_{14} - \vec{b}_{23}) \quad (22)$$

under \vec{H}' . We can now see that these are the roots of the $B_2 = O(2, 3)$ algebra, where the simple roots can be taken to be

$$\vec{\alpha}_1 = \vec{b}_{12} , \quad \vec{\alpha}_2 = \frac{1}{2}(\vec{b}_{24} - \vec{b}_{13}) . \quad (23)$$

The generators $E_{\vec{a}_{ijk}}$ form a 4-dimensional spinor representation under $O(2,3)$. The highest-weight generator is $E_{\vec{a}_{124}}$, which has weight $\frac{1}{2}(\vec{a}_{124} - \vec{a}_{123})$ under \vec{H}' . Thus the coset of the $SL(5, \mathbb{R})$ orbit of this pair of weights is given by

$$\frac{SL(5, \mathbb{R})}{O(2,3) \ltimes \mathbb{R}^4} . \quad (24)$$

This has dimensions 10, which is the same as the dimension of the total charge space for 2-forms in $D = 7$. As usual, by looking at the action of the $SL(5, \mathbb{R})$ generators on the two states $|\vec{a}_{23}\rangle$ and $|\vec{b}_3\rangle$, we can verify that indeed all generic points in the 10-dimensional charge space can be reached. The generic ten-dimensional 2-charge orbits preserve $\frac{1}{4}$ of the supersymmetry, while the degenerate seven-dimensional single-charge orbits preserve $\frac{1}{2}$.

5 $D = 6$

The global the symmetry group in $D = 6$ is $O(5,5)$, for which the simple roots are $\vec{b}_{i,i+1}$, with $i = 1, 2, 3, 4$, and \vec{a}_{123} . The five 3-forms $F_3^{(i)}$, together with their duals, form a 10-dimensional irreducible representation of $O(5,5)$, with weights $(\vec{a}_i, -\vec{a}_i)$. The highest-weight vector is $-\vec{a}_1$. Splitting the internal index i , which runs over 5 values, as $i = (1, \alpha)$, we easily see that the following subset of the $O(5,5)$ generators annihilate the state $|\vec{a}_1\rangle$:

$$\begin{aligned} E_{\pm \vec{b}_{\alpha\beta}} , \quad E_{\pm \vec{a}_{1\alpha\beta}} , \quad E_{\vec{b}_{1\alpha}} , \quad E_{\vec{a}_{\alpha\beta\gamma}} , \\ \vec{H}' = \vec{H} - \frac{1}{2}(\vec{a}_1 \cdot \vec{H}) \vec{a}_1 , \end{aligned} \quad (25)$$

while the remaining generators give

$$E_{-\vec{b}_{1\alpha}} |\vec{a}_1\rangle = |-\vec{a}_\alpha\rangle , \quad E_{-\vec{a}_{\alpha\beta\gamma}} |\vec{a}_1\rangle = |\vec{a}_\delta\rangle . \quad (26)$$

where in the last case the indices α, β, γ and δ , which take their values in the set $(2,3,4,5)$, are all different. The conjugate pairs of generators in (25) have weights

$$\pm \vec{b}_{\alpha\beta} , \quad \pm \vec{a}_{1\alpha\beta} , \quad \vec{b}_{1\alpha} , \quad \vec{a}_{\alpha\beta\gamma} , \quad (27)$$

under \vec{H}' , and they are precisely the root vectors of $D_4 = O(4,4)$. The simple roots may be taken to be

$$\vec{\alpha}_1 = \vec{b}_{23} , \quad \vec{\alpha}_2 = \vec{b}_{34} , \quad \vec{\alpha}_3 = \vec{b}_{45} , \quad \vec{\alpha}_4 = \vec{a}_{123} . \quad (28)$$

The remaining eight generators $E_{\vec{b}_{1\alpha}}$ and $E_{\vec{a}_{\alpha\beta\gamma}}$ in (25) form an 8-dimensional representation under $O(4,4)$. The highest-weight state is $E_{\vec{a}_{345}}$, with weight $\vec{a}_{345} + \vec{a}_1 = -\vec{a}_2$ under \vec{H}' . Thus the coset parameterising the single-charge 3-form solutions in $D = 6$ is

$$\frac{O(5,5)}{O(4,4) \ltimes \mathbb{R}^8} . \quad (29)$$

There is again an easier explanation for the denominator group that leaves a single 3-form charge invariant. In $D = 6$, there is an $O(4, 4)$ T-duality, under which the NS-NS 3-form field strength $F_3^{(1)}$ is a singlet. In addition, the eight R-R axions $\mathcal{F}_1^{(1\alpha)}$, $F_1^{(\alpha\beta\gamma)}$, which can all be covered by derivatives simultaneously, form an 8-dimensional spinor representation under the T-duality group. In fact $O(4, 4) \ltimes \mathbb{R}^8$ is precisely the global symmetry of the version of the supergravity obtained by dualising only the higher-degree R-R fields [13].

The coset (29) has dimension 9, which is one less than the dimension of the charge-vector space for 3-forms. In fact, the denominator group in (29) is the stability group of a light-like vector in $O(5, 5)$. We shall return to this point below. For now, let us proceed by extending the discussion to 2-charge solutions. These are in fact dyonic (of the first kind, in the notation of [5]), where a single field strength carries both electric and magnetic charges [14]. To study the 2-charge orbits, we may therefore consider the stability group for the pair of states $|\vec{a}_1\rangle$ and $|\vec{-a}_1\rangle$. It is easily established that their sum is left invariant by the subset of $O(5, 5)$ generators

$$E_{\pm\vec{b}_{\alpha\beta}} \ , \quad E_{\pm\vec{a}_{1\alpha\beta}} \ , \quad (E_{\pm\vec{a}_{\alpha\beta\gamma}} - E_{\mp\vec{b}_{1\delta}}) \ , \quad (30)$$

where in the last case α, β, γ and δ are all different, together with the same set of four Cartan generators \vec{H}' given in (25). (Since the weights \vec{a}_1 and $-\vec{a}_1$ are parallel, there are still four combinations of the $O(5, 5)$ Cartan generators that leave the two states invariant.) The generators in (30) have weights

$$\pm\vec{b}_{\alpha\beta} \ , \quad \pm\vec{a}_{1\alpha\beta} \ , \quad \pm\frac{1}{2}(\vec{a}_{\alpha\beta\gamma} - \vec{b}_{1\delta}) \quad (31)$$

under \vec{H}' ; these are precisely the roots of $B_4 = O(4, 5)$. The simple roots may be taken to be

$$\vec{\alpha}_1 = \vec{b}_{23} \ , \quad \vec{\alpha}_2 = \vec{b}_{34} \ , \quad \vec{\alpha}_3 = \vec{b}_{45} \ , \quad \vec{\alpha}_4 = \frac{1}{2}(\vec{a}_{234} - \vec{b}_{15}) \ . \quad (32)$$

Thus the coset parameterising the 2-charge 3-form solutions is

$$\frac{O(5, 5)}{O(4, 5)} \ . \quad (33)$$

Note that in this case all the generators that annihilate the states lie in the single simple group $O(4, 5)$, and there are no additional \mathbb{R} factors.

The denominator group in the 2-charge coset is the stability group for a timelike or spacelike vector in $O(5, 5)$, and indeed the coset (33) has dimension 9. This is what one expects for timelike or spacelike 10-vectors of fixed length. The rescaling of the length can be implemented by using the trombone symmetry of the $D = 6$ equations of motion. By

including this rescaling, all points in the 10-dimensional charge space for 3-forms in $D = 6$ correspond to valid string solutions. There are three independent orbits; timelike, spacelike and null. To understand this, we note that the electric charges Q_i and the magnetic charges P_i assemble into the vector representation of $O(5,5)$ in the form $(Q_i + P_i, Q_i - P_i)$. The magnitude of this vector is

$$\sum_i (Q_i + P_i)^2 - \sum_i (Q_i - P_i)^2 = 4 \sum_i Q_i P_i . \quad (34)$$

Thus we see that the charge vector is null on the orbit of any single-charge solution, and timelike or spacelike on any of the 2-charge orbits. Note that the null orbits preserve $\frac{1}{2}$ the supersymmetry, while the timelike and spacelike orbits preserve $\frac{1}{4}$. It is worth remarking that the self-dual string and the anti-self-dual string belong to different multiplets under $O(5,5)$, with the former lying on the spacelike orbit, and the latter on the timelike orbit.

Another way to understand why the dyonic strings with different relative signs of their electric and magnetic charges belong to different multiplets is the following. The weight vectors for the electric and magnetic charges are anti-parallel, and hence there is only a single Cartan generator that can rescale the charges, and therefore it cannot change the relative sign of the two charges. As we shall see in the next two sections, the weight vectors for simple 3-charge solutions in $D = 5$ and for simple 4-charge solutions in $D = 4$ are again degenerate, in that they lie in 2-dimensional and 3-dimensional planes respectively. Thus the number of Cartan generators that can rescale the charges is one less than the number of charges. It follows that in both these cases, configurations with different relative signs of the charges belong to different multiplets. It is interesting to note that this phenomenon always arises in the cases where the dilatonic scalar fields in the solution are regular, even at the horizon.

In the fully-dualised six-dimensional theory there are in total sixteen 2-forms, namely $F_2^{(ij)}$, $\mathcal{F}_2^{(i)}$ and $*F_4$. Their dilaton vectors \vec{a}_{ij} , \vec{b}_i and $-\vec{a}$ form an irreducible 16-dimensional representation of $O(5,5)$, with $-\vec{a}$ as the highest weight. The state $|- \vec{a}\rangle$ is annihilated by the following subset of the $O(5,5)$ generators:

$$\begin{aligned} E_{\pm \vec{b}_{ij}} , \quad E_{\vec{a}_{ijk}} , \\ \vec{H}' = \vec{H} - \frac{2}{5}(\vec{a} \cdot \vec{H}) \vec{a} , \end{aligned} \quad (35)$$

while the generators $E_{-\vec{a}_{ijk}}$ give

$$E_{-\vec{a}_{ijk}} |- \vec{a}\rangle = |\vec{a}_{\ell m}\rangle , \quad (36)$$

where i, j, k, ℓ and m are all different. The conjugate pairs of generators in (35) have weights $\pm \vec{b}_{ij}$ under \vec{H}' ; these are the roots of the algebra $SL(5, \mathbb{R})$, for which the simple roots may be taken to be

$$\vec{\alpha}_1 = \vec{b}_{12} , \quad \vec{\alpha}_2 = \vec{b}_{23} , \quad \vec{\alpha}_3 = \vec{b}_{34} , \quad \vec{\alpha}_4 = \vec{b}_{45} . \quad (37)$$

The remaining generators $E_{\vec{a}_{ijk}}$ in (35) mutually commute, and form a 10-dimensional representation under $SL(5, \mathbb{R})$. The highest-weight generator is $E_{\vec{a}_{345}}$, which has weight $\vec{a}_{345} + \frac{4}{5}\vec{a}$ under \vec{H}' . Therefore, the coset parameterising the single-charge 2-form solutions in $D = 6$ is

$$\frac{O(5, 5)}{SL(5, \mathbb{R}) \ltimes \mathbb{R}^{10}} . \quad (38)$$

Note that the $SL(5, \mathbb{R})$ can be understood as being inherited directly from the global symmetry group in $D = 7$. In particular, if we were to consider a single-charge solution in $D = 6$ using the new Kaluza-Klein 2-form $\mathcal{F}_2^{(5)}$ rather than $*F_4$, then the $SL(5, \mathbb{R})$ of $D = 7$ would leave it invariant. Also, the constant shift symmetries \mathbb{R}^{10} of the ten axions coming from the dimensional reduction of the ten vectors in $D = 7$ also leave the new Kaluza-Klein vector invariant. Since the solution using $\mathcal{F}_2^{(5)}$ in $D = 6$ is equivalent, under $O(5, 5)$, to the solution using $*F_4$, the stability group will be the same (modulo conjugations) in each case.

The dimension of the coset (38) for single-charge 2-form solutions in $D = 6$ is 11, while the total charge-space for 2-forms has dimension 16. General points in the charge space can be reached by looking at the orbits of 2-charge solutions. We shall start from the simple 2-charge solution supported by $*F_4$ and $\mathcal{F}_2^{(5)}$, corresponding to the sum of the states $|- \vec{a}\rangle$ and $|\vec{b}_5\rangle$. We find that this sum of states is annihilated by the following subset of $O(5, 5)$ generators:

$$E_{\pm \vec{b}_{ab}} , \quad (E_{\pm \vec{a}_{ab5}} - E_{\mp \vec{a}_{cd5}}) , \quad E_{-\vec{b}_{a5}} , \quad E_{\vec{a}_{abc}} , \quad (39)$$

$$\vec{H}' = \vec{H} + \frac{1}{8}(3\vec{a} \cdot \vec{H} - 5\vec{b}_5 \cdot \vec{H})\vec{b}_5 + \frac{1}{8}(3\vec{b}_5 \cdot \vec{H} - 5\vec{a} \cdot \vec{H})\vec{a} \quad (40)$$

where we have split $i = (a, 5)$, and in the second case a, b, c and d are all different. The conjugate pairs of generators have weights

$$\pm \vec{b}_{ab} , \quad \pm \frac{1}{2}(\vec{a}_{ab5} - \vec{a}_{cd5}) \quad (41)$$

under \vec{H}' ; these are the roots of the algebra $B_3 = O(3, 4)$. The simple roots may be taken to be

$$\vec{\alpha}_1 = \vec{b}_{12} , \quad \vec{\alpha}_2 = \vec{b}_{23} , \quad \vec{\alpha}_3 = \frac{1}{2}(\vec{a}_{145} - \vec{a}_{235}) . \quad (42)$$

The further eight generators $E_{-\vec{b}_{a5}}$ and $E_{\vec{a}_{abc}}$ in (39) mutually commute, and form a spinor representation under $O(3, 4)$. The highest-weight generator is $E_{-\vec{b}_{45}}$, which has weight

$-\vec{b}_{45} - \frac{5}{4}\vec{b}_5 + \frac{3}{4}\vec{a}$. The coset parameterising the 2-charge 2-form solutions in $D = 6$ is therefore

$$\frac{O(5,5)}{O(3,4) \ltimes \mathbb{R}^8} . \quad (43)$$

This has dimension 16, which is the same as the total dimension of the charge space for 2-forms in $D = 6$. Indeed, one can reach all generic points in the charge vector space, by starting from a simple 2-charge. The degenerate eleven-dimensional single-charge orbits correspond as usual to solutions that preserve $\frac{1}{2}$ the supersymmetry, while the generic sixteen-dimensional 2-charge orbit preserves $\frac{1}{4}$.

6 $D = 5$

The global symmetry group for the fully-dualised theory in $D = 5$ is E_6 . The simple roots are $\vec{b}_{i,i+1}$, with $i = 1, 2, 3, 4, 5$, and \vec{a}_{123} . The full set of roots is $\pm\vec{b}_{ij}$, $\pm\vec{a}_{ijk}$, and $\mp\vec{a}$. In the fully-dualised theory, there are only 2-form field strengths (in addition to the axions), namely $\{F_2^{(ij)}, \mathcal{F}_2^{(i)}, *F_3^{(i)}\}$. Their dilaton vectors $\{\vec{a}_{ij}, \vec{b}_i, -\vec{a}_i\}$ are the weights of the 27-dimensional irreducible representation of E_6 , with $-\vec{a}_1$ as the highest weight. The associated state $|- \vec{a}_1\rangle$ is annihilated by the subset of E_6 generators

$$E_{\pm\vec{b}_{\alpha\beta}} , \quad E_{\pm\vec{a}_{1\alpha\beta}} , \quad E_{\vec{b}_{1\alpha}} , \quad E_{\vec{a}_{\alpha\beta\gamma}} , \quad E_{-\vec{a}} , \quad (44)$$

$$\vec{H}' = \vec{H} - \frac{3}{8}(\vec{a}_1 \cdot \vec{H}) \vec{a}_1 , \quad (45)$$

where we have split $i = (1, \alpha)$. The conjugate pairs of generators have weights

$$\pm \vec{b}_{\alpha\beta} , \quad \pm a_{1\alpha\beta} \quad (46)$$

under \vec{H}' , and thus form the roots of the algebra $D_5 = O(5,5)$. The simple roots can be taken to be

$$\vec{\alpha}_1 = \vec{b}_{23} , \quad \vec{\alpha}_2 = \vec{b}_{34} , \quad \vec{\alpha}_3 = \vec{b}_{45} , \quad \vec{\alpha}_4 = \vec{b}_{56} , \quad \vec{\alpha}_5 = \vec{a}_{123} . \quad (47)$$

The remaining generators $E_{\vec{b}_{1\alpha}}$, $E_{\vec{a}_{\alpha\beta\gamma}}$ and $E_{-\vec{a}}$ in (44) are mutually-commuting, and form a 16-dimensional spinor representation of $O(5,5)$. The highest-weight generator is $E_{-\vec{a}}$, with weight $-\vec{a} + \frac{3}{4}\vec{a}_1$ under the $O(5,5)$ Cartan generators \vec{H}' . The coset parameterising the single-charge 2-form solutions in $D = 5$ is therefore

$$\frac{E_6}{O(5,5) \ltimes \mathbb{R}^{16}} . \quad (48)$$

There are two more intuitive ways to understand the denominator group. One is to observe that the NS-NS 3-form $F_3^{(1)}$ is a singlet under the T-duality group $O(5,5)$, and it is also

invariant under the \mathbb{R}^{16} shift symmetries of the sixteen R-R axions $\{\mathcal{F}_1^{(1\alpha)}, F_1^{(\alpha\beta\gamma)}, *F_4\}$. Another way to understand the denominator group is to view it as being inherited from $D = 6$, which obviously leaves the new Kaluza-Klein vector $\mathcal{F}_2^{(6)}$ invariant. The \mathbb{R}^{16} in this case is generated by the commuting shift symmetries of the axions coming from the dimensional reduction of the sixteen 2-forms in $D = 6$. Since the new Kaluza-Klein vector is equivalent, under E_6 , to the 2-form $*F_3^{(1)}$ that we were previously considering, it follows that the stability group will also be the same, modulo conjugations.

Moving now to 2-charge 2-form solutions in $D = 5$, we may take these to be represented by the states $|- \vec{a}_1\rangle$ and $|\vec{a}_{16}\rangle$. This combination is left invariant by the subset of E_6 generators

$$\begin{aligned} & E_{\pm \vec{b}_{ab}} , \quad E_{\pm \vec{a}_{1ab}} , \quad (E_{\pm \vec{a}_{abc}} - E_{\mp \vec{b}_{1d}}) , \\ & E_{\vec{b}_{16}} , \quad E_{\vec{b}_{a6}} , \quad E_{\vec{a}_{1a6}} , \quad E_{\vec{a}_{a66}} , \quad E_{-\vec{a}} , \\ & \vec{H}' = \vec{H} + \frac{1}{4}(\vec{a}_1 \cdot \vec{H} - 2\vec{a}_{16} \cdot \vec{H}) \vec{a}_{16} + \frac{1}{4}(\vec{a}_{16} \cdot \vec{H} - 2\vec{a}_1 \cdot \vec{H}) \vec{a}_1 , \end{aligned} \quad (49)$$

where we have split the internal indices as $i = (1, a, 6)$, and in the final expression on the top line, a, b, c and d are all different. We find that the conjugate pairs of generators have weights

$$\pm \vec{b}_{ab} , \quad \pm \vec{a}_{1ab} , \quad \pm \frac{1}{2}(\vec{a}_{abc} - \vec{b}_{1d}) \quad (50)$$

under \vec{H}' . These are the roots of the algebra $B_4 = O(4, 5)$, for which we may take the simple roots to be

$$\vec{\alpha}_1 = \vec{b}_{23} , \quad \vec{\alpha}_2 = \vec{b}_{34} , \quad \vec{\alpha}_3 = \vec{b}_{45} , \quad \vec{\alpha}_4 = \frac{1}{2}(\vec{a}_{234} - \vec{b}_{15}) . \quad (51)$$

The remaining generators in (49) mutually commute, and form a 16-dimensional irreducible spinor representation of $O(4, 5)$. The highest weight generator is $E_{-\vec{a}}$, with weight $-\frac{1}{2}(\vec{b}_{16} + \vec{a})$ under \vec{H}' . The coset parameterising the 2-charge 2-form solutions in $D = 5$ is therefore

$$\frac{E_6}{O(4, 5) \ltimes \mathbb{R}^{16}} . \quad (52)$$

The denominator group can in fact also be understood from the observation that the 2-charge solution using $\{*F_3^{(1)}, F_2^{(16)}\}$ is the dimensional reduction of the dyonic string in $D = 6$, supported by electric and magnetic charges for $F_3^{(1)}$. This field strength was already previously found to be invariant under $O(4, 5)$. The commuting \mathbb{R}^{16} symmetries are the shift symmetries of the sixteen axions coming from the dimensional reduction of the sixteen 2-forms in $D = 6$.

The dimension of the coset (52) for 2-charge 2-forms in $D = 5$ is 26, which is one less than the total dimension of the charge vector space for 2-forms. Thus we see that not all points in the vector space can be reached as the orbits of a simple 2-charge solution. In particular, starting from the weight vectors $-\vec{a}_1$ and \vec{a}_{16} , we cannot generate the weight vector \vec{b}_6 . However, there exist simple 3-charge solutions in $D = 5$, precisely using the field strengths $\{*F_3^{(1)}, F_2^{(16)}, \mathcal{F}_2^{(6)}\}$ [5]. Now, we may therefore augment the previous discussion by considering as a starting point the 3-charge configuration represented as the sum of states $|-\vec{a}_1\rangle$, $|\vec{a}_{16}\rangle$ and $|\vec{b}_6\rangle$. It is easy to check as usual from the dilaton vector sum rules that this combination of states is annihilated by the subset of E_6 generators

$$\begin{aligned} E_{\pm\vec{b}_{ab}} , \quad E_{\pm\vec{a}_{1ab}} , \quad (E_{\pm\vec{b}_{1a}} - E_{\mp\vec{a}_{bcd}}) , \\ (E_{\pm\vec{a}_{ab6}} - E_{\mp\vec{a}_{cd6}}) , \quad (E_{\pm\vec{b}_{a6}} - E_{\mp\vec{a}_{1a6}}) , \quad (E_{\pm\vec{b}_{16}} - E_{\mp\vec{a}}) , \end{aligned} \quad (53)$$

where as before we have split the internal indices as $i = (1, a, 6)$, together with the same subset of Cartan generators \vec{H}' that is defined in (49). (The reason why there are still four combinations of the E_6 Cartan generators that annihilate the three states in this 3-charge case is that the third charge has a dilaton vector that is coplanar with the first two.) It is straightforward to verify that the full set of generators in (53) have the weights

$$\begin{aligned} \pm\vec{b}_{ab} , \quad \pm\vec{a}_{1ab} , \quad \pm\frac{1}{2}(\vec{b}_{1a} - \vec{a}_{bcd}) , \\ \pm\frac{1}{2}(\vec{a}_{ab6} - \vec{a}_{cd6}) , \quad \pm\frac{1}{2}(\vec{b}_{a6} - \vec{a}_{1a6}) , \quad \pm\frac{1}{2}(\vec{b}_{16} + \vec{a}) \end{aligned} \quad (54)$$

under \vec{H}' , and that they are nothing but the roots of the F_4 algebra. The simple roots can be taken to be

$$\vec{\alpha}_1 = \vec{b}_{34} , \quad \vec{\alpha}_2 = \vec{b}_{45} , \quad \vec{\alpha}_3 = \frac{1}{2}(\vec{a}_{234} - \vec{b}_{15}) , \quad \vec{\alpha}_4 = \frac{1}{2}(\vec{b}_{16} + \vec{a}) . \quad (55)$$

The coset parameterising the 3-charge 2-form solutions in $D = 5$ is therefore

$$\frac{E_6}{F_4} . \quad (56)$$

The fact that the denominator group F_4 contains $O(4, 5)$ as a subgroup can be understood from the fact that the 3-charge solution becomes a boosted dyonic string, *i.e.* an intersection of a pp wave and a dyonic string, in $D = 6$, whose charges are obviously invariant under the $O(4, 5)$ stability group that we found in section 5.

The dimension of the 3-charge coset space (56) is 26, as is the dimension of the 2-charge coset space (52). In fact the 2-charge orbits correspond to null vectors in the 27-dimensional charge vector space, while the 3-charge orbits correspond to timelike or spacelike vectors [11].

The reason why these 3-charge orbits have only dimension 26 is that an overall scaling symmetry, *i.e.* the trombone symmetry, is also needed in order to fill out the entire 27-dimensional charge vector space. This can be seen from the fact that the three weight vectors $\{-\vec{a}_1, \vec{a}_{16}, \vec{b}_6\}$ associated with the three charges are coplanar, and so only two, rather than three, E_6 Cartan generator combinations are able to scale the charges. This is analogous to the situation that we saw for dyonic strings in $D = 6$. This situation contrasts with that for the 26-dimensional orbits for the 2-charge solutions in $D = 5$, where, as we saw above it is a non-zero weight, *i.e.* \vec{b}_6 , rather than a scaling of one of the basic charges, whose absence reduces the dimension of the orbits from 27 to 26.

The 3-charge orbits preserve $\frac{1}{8}$ of the supersymmetry, while the 2-charge orbits preserve $\frac{1}{4}$ and the 1-charge orbits preserve $\frac{1}{2}$.

7 $D = 4$

The global symmetry group of the fully-dualised theory in $D = 4$ is E_7 . The simple roots are $\vec{b}_{i,i+1}$ with $1 \leq i \leq 6$, and \vec{a}_{123} . The full set of roots is $\pm\vec{b}_{ij}$, $\pm\vec{a}_{ijk}$ and $\mp\vec{a}_i$. There are 28 2-forms $F_2^{(ij)}$ and $\mathcal{F}_2^{(i)}$ which, together with their duals, form a 56-dimensional irreducible representation of E_7 . The associated dilaton vectors $\{\vec{a}_{ij}, \vec{a}_i, -\vec{a}_{ij}, -\vec{b}_i\}$ are the weights of the 56. The highest weight vector is $-\vec{b}_7$. We find that the associated state $|- \vec{b}_7\rangle$ is annihilated by the following subset of the E_7 generators:

$$\begin{aligned} E_{\pm b_{ab}} , \quad E_{\pm \vec{a}_{abc}} , \quad E_{\mp \vec{a}_7} , \quad E_{\vec{b}_{a7}} , \quad E_{\vec{a}_{ab7}} , \quad E_{-\vec{a}_a} , \\ \vec{H}' = \vec{H} - \frac{1}{3}(\vec{b}_7 \cdot \vec{H}) \vec{b}_7 , \end{aligned} \quad (57)$$

where we have split the internal index as $i = (a, 7)$. We find that the weights of the conjugate pairs of generators under \vec{H}' are given by

$$\pm b_{ab} , \quad \pm \vec{a}_{abc} , \quad \mp \vec{a}_7 , \quad , \quad (58)$$

and that these are precisely the roots of the E_6 algebra. The simple roots may be taken to be

$$\vec{\alpha}_1 = \vec{b}_{12} , \quad \vec{\alpha}_2 = \vec{b}_{23} , \quad \vec{\alpha}_3 = \vec{b}_{34} , \quad \vec{\alpha}_4 = \vec{b}_{45} , \quad \vec{\alpha}_5 = \vec{b}_{56} , \quad \vec{\alpha}_6 = \vec{a}_{123} . \quad (59)$$

The further 27 generators in (57) mutually commute, and form an irreducible 27-dimensional representation under E_6 , with $E_{-\vec{a}_1}$ as the highest-weight generator. It follows that the coset that parameterises the single-charge 2-form solutions in $D = 4$ is

$$\frac{E_7}{E_6 \ltimes \mathbb{R}^{27}} . \quad (60)$$

The denominator group can be easily understood since E_6 is inherited directly from $D = 5$, and the \mathbb{R}^{27} symmetry is generated by the 27 axions that are the dimensional reductions of the 27 2-form field strengths in $D = 5$. The coset (60) has dimension 28. Acting on a single-charge solution with E_7 gives rise to a solution which can carry a maximum of 28 charges. In order to fill out more points in the 56-dimensional charge space, we need to look at orbits involving multi-charge simple solutions.

Simple 2-charge solutions can be constructed using the field strength $\{\mathcal{F}_2^{(7)}, *F_2^{(17)}\}$, associated with the dilaton vectors $\{-\vec{b}_7, -\vec{a}_{17}\}$. We find that the subset of the E_7 generators that leave the combination of states $|- \vec{b}_7\rangle$ and $|- \vec{a}_{17}\rangle$ invariant is

$$\begin{aligned} E_{\pm \vec{b}_{ab}} , \quad E_{\pm \vec{a}_{1ab}} , \quad (E_{\pm \vec{b}_{a7}} - E_{\mp \vec{a}_{1a7}}) , \\ E_{\vec{a}_{abc}} , \quad E_{\vec{a}_{ab7}} , \quad E_{-\vec{a}_1} , \quad E_{-\vec{a}_a} , \quad E_{-\vec{a}_7} , \quad E_{\vec{b}_{1a}} , \quad E_{\vec{b}_{17}} , \end{aligned} \quad (61)$$

$$\vec{H}' = \vec{H} + \frac{1}{8}(\vec{b}_7 \cdot \vec{H} - 3\vec{a}_{17} \cdot \vec{H}) \vec{a}_{17} + \frac{1}{8}(\vec{a}_{17} \cdot \vec{H} - 3\vec{b}_7 \cdot \vec{H}) \vec{b}_7 , \quad (62)$$

where we have split the index $i = (1, a, 7)$. We find that the conjugate pairs of generators in (61) have weights

$$\pm \vec{b}_{ab} , \quad \pm \vec{a}_{1ab} , \quad \pm \frac{1}{2}(\vec{b}_{a7} - \vec{a}_{1a7}) , \quad (63)$$

under H' , and that these are precisely the roots of the $B_5 = O(5, 6)$ algebra. The simple roots can be taken to be

$$\vec{\alpha}_1 = \vec{b}_{23} , \quad \vec{\alpha}_2 = \vec{b}_{34} , \quad \vec{\alpha}_3 = \vec{b}_{45} , \quad \vec{\alpha}_4 = \vec{b}_{56} , \quad \vec{\alpha}_5 = \frac{1}{2}(\vec{b}_{67} - \vec{a}_{167}) . \quad (64)$$

The remaining generators in (61), of which there are 33 in total, form a 32-dimensional spinor representation under $O(5, 6)$ together with a singlet, corresponding to the generator $E_{-\vec{a}_1}$. The highest-weight generator in the 32 is $E_{-\vec{a}_7}$, which has weight $-\vec{a}_7 + \frac{3}{4}\vec{a}_{17} + \frac{1}{4}\vec{b}_7$. Thus the coset for 2-charge 2-form solutions in $D = 4$ is

$$\frac{E_7}{(O(5, 6) \ltimes \mathbb{R}^{32}) \times \mathbb{R}} . \quad (65)$$

This coset has dimension 45.

For 3-charge solutions, we may use the field strengths $\{\mathcal{F}_2^{(7)}, *F_2^{(17)}, F_2^{(16)}\}$, corresponding to the weight vectors $\{-\vec{b}_7, -\vec{a}_{17}, \vec{a}_{16}\}$. We find that the subset of E_7 generators that annihilate this combination of three states is given by

$$\begin{aligned} E_{\pm \vec{b}_{ab}} , \quad E_{\pm \vec{a}_{1ab}} , \quad (E_{\pm \vec{b}_{1a}} - E_{\mp \vec{a}_{bcd}}) , \quad (E_{\pm \vec{b}_{17}} - E_{\pm \vec{a}_6}) , \quad (E_{\pm \vec{b}_{a7}} - E_{\mp \vec{a}_{1a7}}) , \\ (E_{\pm \vec{a}_{ab7}} - E_{\mp \vec{a}_{cd7}}) , \quad E_{\vec{b}_{16}} , \quad E_{\vec{b}_{a6}} , \quad E_{\vec{a}_{1a6}} , \quad E_{\vec{a}_{ab6}} , \\ E_{\vec{a}_{a67}} , \quad E_{-\vec{a}_a} , \quad E_{-\vec{a}_7} , \quad (E_{\vec{a}_{167}} - E_{-\vec{b}_{67}}) , \quad (E_{-\vec{a}_1} - E_{-\vec{b}_{67}}) , \end{aligned} \quad (66)$$

$$\begin{aligned}\vec{H}' &= \vec{H} + \frac{1}{4}(\vec{a}_{16} - \vec{a}_{17} - 2\vec{b}_7) \cdot \vec{H} \vec{b}_7 + \frac{1}{4}(\vec{a}_{16} - 2\vec{a}_{17} - \vec{b}_7) \cdot \vec{H} \vec{a}_{17} \\ &\quad + \frac{1}{4}(-2\vec{a}_{16} + \vec{a}_{17} + \vec{b}_7) \cdot \vec{H} \vec{a}_{16} ,\end{aligned}$$

where we have split the internal indices as $i = (1, a, 6, 7)$. We find that the weights of the conjugate pairs of generators in (66) under \vec{H}' are

$$\begin{aligned}&\pm \vec{b}_{ab} , \quad \pm \vec{a}_{1ab} , \quad \pm \frac{1}{2}(\vec{b}_{1a} - \vec{a}_{bcd}) , \quad \pm \frac{1}{2}(\vec{b}_{17} + \vec{a}_6) , \\ &\pm \frac{1}{2}(\vec{b}_{a7} - \vec{a}_{1a7}) , \quad \pm \frac{1}{2}(\vec{a}_{ab7} - \vec{a}_{cd7}) ,\end{aligned}\tag{67}$$

and that these coincide with the roots of the F_4 algebra. The simple roots may be taken to be

$$\vec{\alpha}_1 = \vec{b}_{34} , \quad \vec{\alpha}_2 = \vec{b}_{45} , \quad \vec{\alpha}_3 = \frac{1}{2}(\vec{a}_{234} - \vec{b}_{15}) , \quad \vec{\alpha}_4 = \frac{1}{2}(\vec{b}_{17} + \vec{a}_6) .\tag{68}$$

The remaining generators in (66) form a 26-dimensional representation under F_4 , with $E_{\vec{a}_{267}}$ as the highest-weight generator, with weight $\frac{1}{2}(\vec{a}_{267} + \vec{a}_2)$. Thus the coset parameterising the 3-charge 2-form solutions in $D = 4$ is

$$\frac{E_7}{F_4 \ltimes \mathbb{R}^{26}} .\tag{69}$$

Note that the last two generators listed in (66) have zero weight under the Cartan generators \vec{H}' of F_4 ; this can be seen from the fact that these two vectors have components only in the plane orthogonal to the 4-dimensional space spanned by the F_4 roots. In fact, they are the two zero weights in the 26-dimensional representation of F_4 .

The dimension of the coset (69) is 55, which is one less than the dimension of the charge vector space for 2-forms in $D = 4$. In particular, if we start with the three charges corresponding to the weight vectors $\{-\vec{b}_7, -\vec{a}_{17}, \vec{a}_{16}\}$, we can never generate the charge associated with the weight vector \vec{b}_6 by acting with the E_7 global symmetry group. In fact there exists a simple 4-charge solution that uses precisely this extra charge in addition to the previous three. Thus we may now take this 4-charge configuration as a new starting point, and examine its orbit under E_7 . We find that the corresponding combination of four states is annihilated by the following subset of E_7 generators

$$\begin{aligned}&E_{\pm \vec{b}_{ab}} , \quad E_{\pm \vec{a}_{1ab}} , \quad (E_{\pm \vec{b}_{1a}} - E_{\mp \vec{a}_{bcd}}) , \\ &(E_{\pm \vec{a}_{ab6}} - E_{\mp \vec{a}_{cd6}}) , \quad (E_{\pm \vec{b}_{a6}} - E_{\mp \vec{a}_{1a6}}) , \quad (E_{\pm \vec{b}_{16}} - E_{\pm \vec{a}_7}) , \\ &\vec{H}' = \vec{H} + \frac{1}{4}(\vec{a}_{16} - \vec{a}_{17} - 2\vec{b}_7) \cdot \vec{H} \vec{b}_7 + \frac{1}{4}(\vec{a}_{16} - 2\vec{a}_{17} - \vec{b}_7) \cdot \vec{H} \vec{a}_{17} \\ &\quad + \frac{1}{4}(-2\vec{a}_{16} + \vec{a}_{17} + \vec{b}_7) \cdot \vec{H} \vec{a}_{16}\end{aligned}\tag{70}$$

and

$$(E_{\pm\vec{b}_{17}} - E_{\pm\vec{a}_6}) , \quad (E_{\pm\vec{b}_{a7}} - E_{\mp\vec{a}_{1a7}}) , \quad (E_{\pm\vec{a}_{ab7}} - E_{\mp\vec{a}_{cd7}}) , \quad (E_{\pm\vec{a}_{a67}} - E_{\pm\vec{a}_a}) , \quad (71)$$

$$H_5 = E_{\vec{b}_{67}} + E_{-\vec{b}_{67}} - E_{\vec{a}_{167}} - E_{-\vec{a}_{167}} , \quad H_6 = E_{\vec{b}_{67}} + E_{-\vec{b}_{67}} - E_{\vec{a}_1} - E_{-\vec{a}_1} , \quad (72)$$

where we have again split the internal index as $i = (1, a, 6, 7)$. It is easy to see that the generators in (70) form an F_4 algebra, for which the non-zero roots, measured by \vec{H}' , are

$$\begin{aligned} & \pm\vec{b}_{ab} , \quad \pm\vec{a}_{1ab} , \quad \pm\frac{1}{2}(\vec{b}_{1a} - \vec{a}_{bcd}) , \\ & \pm\frac{1}{2}(\vec{a}_{ab6} - \vec{a}_{cd6}) , \quad \pm\frac{1}{2}(\vec{b}_{a6} - \vec{a}_{1a6}) , \quad \pm\frac{1}{2}(\vec{b}_{16} + \vec{a}_7) . \end{aligned} \quad (73)$$

The simple roots may be taken to be

$$\vec{\alpha}_1 = \vec{b}_{34} , \quad \vec{\alpha}_2 = \vec{b}_{45} , \quad \vec{\alpha}_3 = \frac{1}{2}(\vec{a}_{234} - \vec{b}_{15}) , \quad \vec{\alpha}_4 = \frac{1}{2}(\vec{b}_{16} + \vec{a}_7) . \quad (74)$$

The 26 generators in (71) and (72) form an irreducible 26-dimensional representation under the F_4 . In particular, H_5 and H_6 are the two states with zero weights under \vec{H}' . The weights of the remaining 24 generators (71) under \vec{H}' are given by

$$\pm\frac{1}{2}(\vec{b}_{17} + \vec{a}_6) , \quad \pm\frac{1}{2}(\vec{b}_{a7} - \vec{a}_{1a7}) , \quad \pm\frac{1}{2}(\vec{a}_{ab7} - \vec{a}_{cd7}) , \quad \pm\frac{1}{2}(\vec{a}_{a67} + \vec{a}_a) . \quad (75)$$

In fact the full set of all the generators in (70), (71) and (72) generate an E_6 algebra, with H_5 and H_6 giving the extra two Cartan generators over and above the four Cartan generators \vec{H}' of F_4 . Thus we have a description of E_6 in terms of its decomposition under F_4 , with $78 \rightarrow 52 + 26$. Note that the extra Cartan generators H_5 and H_6 are compact, unlike the standard non-compact Cartan generators that have arisen previously, and so the version of E_6 that we obtain here is $E_{6(2)}$ rather than the maximally non-compact form $E_{6(6)}$. Thus the coset parameterising the 4-charge 2-form solutions in $D = 4$ is

$$\frac{E_7}{E_{6(2)}} . \quad (76)$$

This result was also obtained in [11].

Before discussing the interpretation of these results for BPS 2-form solutions, we should first note that there is a new feature in $D = 4$, namely that there also exists another class of charge configuration, which is associated with extremal solutions that are not supersymmetric. Specifically, there exists a 2-charge dyonic solution, where the electric and magnetic charges are carried by the same field strength. Thus, for example, we may consider the electric and magnetic charges associated with the field strength $\mathcal{F}_2^{(7)}$. We should therefore look

for the stability subgroup of E_7 that leaves the combination of states $|- \vec{b}_7\rangle$ and $|\vec{b}_7\rangle$ invariant. We find that the following subset of the E_7 generators annihilate this combination of states:

$$E_{\pm \vec{b}_{ab}} , \quad E_{\pm \vec{a}_{abc}} , \quad E_{\mp \vec{a}_7} , \\ \vec{H}' = \vec{H} - \frac{1}{3}(\vec{b}_7 \cdot \vec{H}) \vec{b}_7 , \quad (77)$$

where $i = (a, 7)$. We easily find that these are the generators of E_6 , with a regular embedding in E_7 . The weights of the E generators under \vec{H}' are

$$\pm \vec{b}_{ab} , \quad \pm \vec{a}_{abc} , \quad \mp \vec{a}_7 . \quad (78)$$

The simple roots may be taken to be

$$\vec{\alpha}_1 = \vec{b}_{12} , \quad \vec{\alpha}_2 = \vec{b}_{23} , \quad \vec{\alpha}_3 = \vec{b}_{34} , \quad \vec{\alpha}_4 = \vec{b}_{45} , \quad \vec{\alpha}_5 = \vec{b}_{56} , \quad \vec{\alpha}_6 = \vec{a}_{123} . \quad (79)$$

The coset parameterising the non-supersymmetric dyonic 2-form solutions in $D = 4$ is therefore given by

$$\frac{E_7}{E_6} , \quad (80)$$

which has dimension 55. Note that the E_6 denominator group here is the standard maximally non-compact version $E_{6(6)}$.

Now we are in a position to discuss the supersymmetry of the various charge multiplets in $D = 4$. The situation for the 1-charge orbits whose coset is given by (60), is straightforward; these solutions all preserve $\frac{1}{2}$ of the supersymmetry. Similarly, the 2-charge orbits with coset given by (65) all preserve $\frac{1}{4}$ of the supersymmetry, and the 3-charge orbits with coset (69) all preserve $\frac{1}{8}$. The situation is more complicated for the 4-charge orbits, where the coset is given by (76). In these cases the solutions either preserve $\frac{1}{8}$ of the supersymmetry, or they preserve no supersymmetry at all, depending upon the relative signs of the four charges. Of the 16 possible signs, 8 lead to $\frac{1}{8}$ supersymmetry, while the other 8 lead to non-supersymmetric solutions. (This phenomenon was first seen in the context of the type IIA string in [6, 15]. An explanation for why it occurs can be found in [4].)

A completely different kind of non-supersymmetric solution arises in the case of the dyonic 2-charge configurations whose orbits are given by (80). Here, the supersymmetry breaking is quite independent of the signs of the charges, and the solutions are not related by any duality symmetries to the non-supersymmetric 4-charge solutions discussed above. This can be seen from the fact that the 4-charge solutions are described by four harmonic functions, implying that although non-supersymmetric, there is still no force between the

charges. By contrast, the 2-charge dyonic solution cannot be expressed in terms of harmonic functions, and indeed there is a repulsive force between the two charges [16].

8 Solutions with 1-form field strengths

So far, we have discussed the structure of the orbits of single-charge and multi-charge p -brane solutions that are supported by 4-form, 3-form or 2-form field strengths. There also exist p -brane solutions that are supported by 1-form field strengths, *i.e.* by axionic scalar fields. The situation is much more complicated for these, because they transform non-linearly under the global symmetry groups of the fully-dualised versions of the supergravity theories. However, different versions of the theories, in which other dualisation possibilities are implemented, can have different global symmetries. For example, if we consider the four-dimensional supergravity obtained by direct dimensional reduction from $D = 11$ without any dualisation, it has an $SL(7, \mathbb{R}) \ltimes \mathbb{R}^{35}$ global symmetry, where the \mathbb{R}^{35} corresponds to the shift symmetries of the 35 axions coming from the dimensional reduction of A_3 in $D = 11$. In this version of $D = 4$ supergravity, the 35 axions form a linear tensor representation under $SL(7, \mathbb{R})$. We shall take this as an example, to illustrate how the stability subgroups depend on the number of charges in 1-form solutions (strings) supported by these axions.

The simple roots for the $SL(7, \mathbb{R})$ algebra are $\vec{b}_{i,i+1}$, for $1 \leq i \leq 6$. The full set of $SL(7, \mathbb{R})$ roots are $\pm \vec{b}_{ij}$. As explained above, we shall restrict our attention to the 35 of 1-form field strengths $F_1^{(ijk)}$ that transform linearly under $SL(7, \mathbb{R})$. Their dilaton vectors \vec{a}_{ijk} form the weights of the 35 of $SL(7, \mathbb{R})$. Let us begin by considering a 1-charge solution corresponding to the state $|\vec{a}_{123}\rangle$. This is annihilated by the following subset of the $SL(7, \mathbb{R})$ generators:

$$\begin{aligned} E_{\pm \vec{b}_{12}} , \quad E_{\pm \vec{b}_{13}} , \quad E_{\pm \vec{b}_{23}} , \quad \vec{E}_{\vec{b}_{ab}} , \\ E_{-\vec{b}_{1a}} , \quad E_{-\vec{b}_{2a}} , \quad E_{-\vec{b}_{3a}} , \end{aligned} \tag{81}$$

$$\vec{H}' = \vec{H} - \frac{1}{4}(\vec{a}_{123} \cdot \vec{H}) \vec{a}_{123} , \tag{82}$$

where we have split the internal index as $i = (1, 2, 3, a)$. It is easy to see that the conjugate pairs of generators, together with the five Cartan generators \vec{H}' , give an $SL(3, \mathbb{R}) \times SL(4, \mathbb{R})$ algebra (with $E_{\pm \vec{b}_{ab}}$ associated with $SL(4, \mathbb{R})$). The non-zero roots are

$$\pm \vec{b}_{12} , \quad \pm \vec{b}_{13} , \quad \pm \vec{b}_{23} , \quad \pm \vec{b}_{ab} . \tag{83}$$

The simple roots may be taken to be

$$\vec{\alpha}_1 = \vec{b}_{12} , \quad \vec{\alpha}_2 = \vec{b}_{23} , \quad \vec{\alpha}_3 = \vec{b}_{45} , \quad \vec{\alpha}_4 = \vec{b}_{56} , \quad \vec{\alpha}_5 = \vec{b}_{67} . \tag{84}$$

The remaining generators in (81), form a mutually-commuting (3,4)-dimensional representation under $SL(3, \mathbb{R}) \times SL(4, \mathbb{R})$. The highest-weight generator is $E_{-\vec{b}_{34}}$, with weight $-\vec{b}_{34}$ under \vec{H}' . Thus the coset describing the 1-charge 1-form solutions is

$$\frac{SL(7, \mathbb{R})}{(SL(3, \mathbb{R}) \times SL(4, \mathbb{R})) \ltimes \mathbb{R}^{12}} . \quad (85)$$

The dimension of this coset is 13, and all points on the orbit preserve $\frac{1}{2}$ of the supersymmetry.

Turning now to 2-charge solutions, we may consider the case where these are associated with the states $|\vec{a}_{123}\rangle$ and $|\vec{a}_{145}\rangle$. We find that the following subset of the $SL(7, \mathbb{R})$ generators annihilate this combination of states:

$$\begin{aligned} & E_{\pm \vec{b}_{23}} , \quad E_{\pm \vec{b}_{45}} , \quad E_{\pm \vec{b}_{67}} , \quad (E_{\pm \vec{b}_{24}} - E_{\mp \vec{b}_{35}}) , \quad (E_{\pm \vec{b}_{25}} - E_{\mp \vec{b}_{34}}) , \\ & E_{-\vec{b}_{12}} , \quad E_{-\vec{b}_{13}} , \quad E_{-\vec{b}_{14}} , \quad E_{-\vec{b}_{15}} , \quad E_{-\vec{b}_{1a}} , \\ & E_{-\vec{b}_{2a}} , \quad E_{-\vec{b}_{3a}} , \quad E_{-\vec{b}_{4a}} , \quad E_{-\vec{b}_{5a}} , \\ & \vec{H}' = \vec{H} - \frac{1}{4}(\vec{a}_{123} \cdot \vec{H}) \vec{a}_{123} - \frac{1}{4}(\vec{a}_{145} \cdot \vec{H}) \vec{a}_{145} , \end{aligned} \quad (86)$$

where we have split the internal index as $i = (1, 2, 3, 4, 5, a)$. The conjugate pairs of generators, together with three of the four Cartan generators \vec{H}' , give the algebra $O(2, 3) \times SL(2, \mathbb{R})$. The non-zero roots under \vec{H}' are

$$\pm \vec{b}_{23} , \quad \pm \vec{b}_{45} , \quad \pm \vec{b}_{67} , \quad \pm \frac{1}{2}(\vec{b}_{24} - \vec{b}_{35}) , \quad \pm \frac{1}{2}(\vec{b}_{25} - \vec{b}_{34}) , \quad (87)$$

and the simple roots may be taken to be

$$\vec{\alpha}_1 = \vec{b}_{23} , \quad \vec{\alpha}_2 = \frac{1}{2}(\vec{b}_{35} - \vec{b}_{24}) , \quad \vec{\alpha}_3 = \vec{b}_{67} . \quad (88)$$

($\vec{\alpha}_3$ is the simple root for $SL(2, \mathbb{R})$.) The remaining Cartan generator in \vec{H}' , orthogonal to the three simple roots, commutes with all the generators in $O(2, 3) \times SL(2, \mathbb{R})$, and enlarges the group to $O(2, 3) \times GL(2, \mathbb{R})$. The remaining generators in (86) form the representations (4,1), (1,2) and (4,2) under $O(2, 3) \times GL(2, \mathbb{R})$. They do not all mutually commute, and the commutator of (4,1) with (4,2) gives (1,2). The coset parameterising the 2-charge 1-form solutions can be written as

$$\frac{SL(7, \mathbb{R})}{(O(2, 3) \times GL(2, \mathbb{R})) \ltimes (R^4 \oplus R^2 \oplus R^8)} \quad (89)$$

We can in principle proceed to study 1-form solutions with larger numbers of charges. In $D = 4$, the maximum number of charges for simple solutions, using the 35 axions $F_1^{(ijk)}$, is 7 [5]. The analysis becomes progressively more complicated as the number of charges increases, and we shall not pursue this further here.

9 Conclusion and discussion

In this paper, we have studied the orbits of the single-charge and multi-charge BPS solutions in maximal supergravities, filled out by acting with the global symmetry groups. By identifying the stability subgroups that leave the original charge configurations fixed, we obtained the coset spaces that parameterise these orbits. In some cases, it turns out that a simple single-charge solution provides an adequate starting point for filling out the entire charge vector space. In other cases, such single-charge starting points have orbits that fail to cover the entire charge vector space. In these cases, simple multi-charge solutions exist, which provide more generic starting points that fill out orbits of larger dimension. By studying these orbits in all dimensions $4 \leq D \leq 9$, we obtained the coset spaces that parameterise the charge vector spaces for all 4-form, 3-form and 2-form solutions. We also looked at the orbits for single-charge and 2-charge solutions supported by 1-form field strengths in $D = 4$, under the $SL(7, \mathbb{R})$ subgroup of E_7 under which the axions coming from the dimensional reduction of A_3 in $D = 11$ form a linear 35-dimensional representation.

The procedure that we have followed in this paper is to investigate how the action of the global symmetry groups of the maximal supergravities allow us to fill out points in the charge vector space by starting from basic single-charge or multi-charge solutions. As we observed in the introduction, this is different from the idea of generating multiplets under a spectrum-generating symmetry that leaves the scalar moduli fixed. As was shown in a previous paper [9], the standard global supergravity symmetry groups can be realised on any single-charge family of solutions while holding the scalar moduli fixed, using a construction in which the moduli are returned to their initial values by compensating transformations constructed using the supergravity symmetry’s Borel subgroup and the “trombone” rescaling transformations, which also act on the metric and thus rescale the masses.⁴ Although this construction is generically non-linear, it nonetheless acts linearly on the charge vectors of single-charge solutions. For a multi-charge solution, exactly the same construction can be carried out, taking any one of the irreducible charge “components” of the multi-charge solution as the basis for the construction. By construction, this will leave the scalar moduli

⁴Since the maximal compact subgroup H of the global symmetry group G leaves the scalar moduli invariant, it might seem more natural to study the orbits of the multi-charge solutions under H . The problem with doing this is that, as discussed in [9], one cannot make the necessary discretisation of H that would be needed in order to generate the discrete orbits of points on the charge lattice that are compatible with Dirac quantisation. The subgroup H plus a trombone symmetry can however be used to generate classical solutions. The coset structure of the orbits of multi-charge black hole solutions under the subgroup H was studied in [17].

fixed and the transformation will act linearly on the chosen irreducible component. Other components of the multi-charge solution will generally be acted on nonlinearly, in particular as a result of the trombone rescaling compensator.

In consequence, the orbits of such fixed-moduli transformations will not in general cover the full charge space in the same way as we have discussed in this paper, where the transformations of the scalar moduli were not taken into account. This point clearly needs more careful study. Here, we shall be content to give a simple example taken from $D = 9$ supergravity, where the standard supergravity symmetry is $GL(2, \mathbb{R})$. One may carry out the construction of [9] for the 2-charge solution (p_1, p_2, q) discussed in section 2, which may be decomposed into irreducible components as $(p_1, p_2, 0) + (0, 0, q)$, and selecting, *e.g.*, the component $(p_1, p_2, 0)$ as the basis for the construction. On this charge-vector component, only the $SL(2, \mathbb{R})$ subgroup of $GL(2, \mathbb{R})$ is genuinely active, and on this component the $SL(2, \mathbb{R})$ transformations act linearly, by construction of [9]. On the $(0, 0, q)$ component, however, which would normally be invariant under the standard $SL(2, \mathbb{R})$, the trombone rescaling part of the transformation will cause a rescaling of q . In the specific case where the scalar moduli are all asymptotically vanishing, one will have $(p_1, p_2, 0) \rightarrow (p'_1, p'_2, 0)$ according to the standard $SL(2, \mathbb{R})$ transformation and $(0, 0, q) \rightarrow (0, 0, \frac{m'}{m}q)$, where $m' = \sqrt{p'^2_1 + p'^2_2}$, $m = \sqrt{p^2_1 + p^2_2}$. Note that the orbits of this transformation do not fill out the complete 2-charge space, unlike the discussion in this paper, which ignores the fixed-modulus constraint.

In conclusion, we may summarise the distinction between orbits of spectrum-generating symmetries and the orbits of charges under the standard supergravity symmetries, ignoring the effect upon the scalar moduli, as follows. For the orbits of simple single-charge solutions, there is no distinction, and so this case is adequately understood. For simple multi-charge solutions, on the other hand, the spectrum-generating orbits will generically cover less of the charge space than those of the standard supergravity symmetries that we have been discussing in this paper. The discrepancy occurs because we currently know only of the single trombone rescaling symmetry that can serve as a compensator to allow the charges to be moved freely while holding the scalar moduli fixed. It is not inconceivable that further symmetries might exist that would allow one to compensate for changes in the integration constants occurring in multiple harmonic functions. If such symmetries were to exist, then the multiplet analysis for multi-charge solutions that we have given could also be interpreted as arising from *bona fide* spectrum-generating symmetries. On the other hand, even if such further symmetries did not exist, the successful treatment of the orbits

for single-charge solutions would provide as complete a description as is possible for the true spectrum-generating orbits. It might in any case be argued to be somewhat artificial to focus attention on the multi-charge solutions where the charges are located at the same point in the transverse space, since the overlapping charges can, at no cost in energy, be separated. Thus the true modulus space for multi-charge solutions is a larger one, in which relative positions and other integration constants are included also. Thus one should really be looking for a symmetry explanation for all the moduli of the multi-charge solutions, and not only those that determine the asymptotic charges and scalar moduli.

Note Added

As the work described in this paper was approaching completion, a paper appeared in which the cosets for BPS solutions in $D = 5$ and $D = 4$ were obtained [11]. The results are substantially in agreement with ours. The approach used in [11] is based on the Jordan algebras associated with the exceptional groups.

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A Cosets for multi-charge p -branes

In this appendix, we present tables that summarise our results for the cosets that parameterise the various single-charge and multi-charge orbits in $4 \leq D \leq 9$. Table 4 contains the results for p -branes supported by 3-form field strengths, and Table 5 contains the results for those supported by 2-forms.

Dimension	$N = 1$	$N = 2$
$D = 9$	$\frac{SL(2, \mathbb{R})}{\mathbb{R}}$	—
$D = 8$	$\frac{SL(3, \mathbb{R})}{SL(2, \mathbb{R}) \ltimes \mathbb{R}^2}$	—
$D = 7$	$\frac{SL(5, \mathbb{R})}{SL(4, \mathbb{R}) \ltimes \mathbb{R}^4}$	—
$D = 6$	$\frac{O(5, 5)}{O(4, 4) \ltimes \mathbb{R}^8}$	$\frac{O(5, 5)}{O(4, 5)}$

Table 4: Cosets for N -charge 3-form solutions

The cosets for single-charge 3-form solutions can all be expressed as $E_{11-D}/(O(10 - D, 10 - D) \ltimes \mathbb{R}^p)$, where $p = 2^{9-D}$. Thus the denominator group can be viewed as the semi-direct product of the T-duality group $O(10 - D, 10 - D)$ with a spinor of $O(10 - D, 10 - D)$. This is not surprising since the NS-NS 3-form field strength is a singlet under T-duality, and is also invariant under the commuting shift symmetries of all the R-R axions, which form a spinor representation under the T-duality group.

Dimension	$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N=2$ dyon
$D = 9$	$\frac{SL(2, \mathbb{R})}{\mathbb{R}}$	$\frac{GL(2, \mathbb{R})}{\mathbb{R}}$	—	—	—
$D = 8$	$\frac{SL(3, \mathbb{R}) \times SL(2, \mathbb{R})}{GL(2, \mathbb{R}) \ltimes \mathbb{R}^3}$	$\frac{SL(3, \mathbb{R}) \times SL(2, \mathbb{R})}{SL(2, \mathbb{R}) \ltimes \mathbb{R}^2}$	—	—	—
$D = 7$	$\frac{SL(5, \mathbb{R})}{(SL(3, \mathbb{R}) \times SL(2, \mathbb{R})) \ltimes \mathbb{R}^6}$	$\frac{SL(5, \mathbb{R})}{O(2, 3) \ltimes \mathbb{R}^4}$	—	—	—
$D = 6$	$\frac{O(5, 5)}{SL(5, \mathbb{R}) \ltimes \mathbb{R}^{10}}$	$\frac{O(5, 5)}{O(3, 4) \ltimes \mathbb{R}^8}$	—	—	—
$D = 5$	$\frac{E_6}{O(5, 5) \ltimes \mathbb{R}^{16}}$	$\frac{E_6}{O(4, 5) \ltimes \mathbb{R}^{16}}$	$\frac{E_6}{F_4}$	—	—
$D = 4$	$\frac{E_7}{E_6 \ltimes \mathbb{R}^{27}}$	$\frac{E_7}{(O(5, 6) \ltimes \mathbb{R}^{32}) \times \mathbb{R}}$	$\frac{E_7}{F_4 \ltimes \mathbb{R}^{26}}$	$\frac{E_7}{E_{6(2)}}$	$\frac{E_7}{E_6}$

Table 5: Cosets for N -charge 2-form solutions

The denominator group for the single-charge 2-form solution in D dimensions is $E_{10-D} \ltimes \mathbb{R}^p$, where p is the number of 2-form field strengths in $D + 1$ dimensions. It is easy to see that this group will leave the new Kaluza-Klein 2-form invariant. Since all the 2-form field strengths in D dimensions are equivalent under E_{11-D} , the coset for the Kaluza-Klein 2-form is the same for any other single-charge solution. The denominator groups for 2-charge solutions also arise with a regular pattern, namely $O(9 - D, 10 - D) \ltimes \mathbb{R}^p$, where $p = 2^{9-D}$, except for $D = 4$, where there is an additional commuting \mathbb{R} symmetry. The \mathbb{R}^p is a spinor under $O(9 - D, 10 - D)$.

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